

VECTOR BUNDLES OVER ORBIT MANIFOLDS

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1. Introduction. Let π denote any finite group, not necessarily abelian, acting freely as diffeomorphisms on some standard sphere S^n . If $M = M^n$ denotes the orbit manifold S^n/π , then it is a compact, connected manifold of dimension n and the quotient map $p: S^n \rightarrow M$ is a principal π -bundle.

If ξ is an orientable real $2r+1$ bundle over M we may interpret it as the homotopy class of a map $\xi: M \rightarrow \text{BSO}(2r+1)$. Now we ask the following questions:

(1.1) when does ξ pull back to a $2s-1$, $1 \leq s \leq r$, bundle over M ?

(1.2) when does ξ pull back to a $2s$, $1 \leq s \leq r$, bundle over M ?

In other words, we want to study the lifting problem

$$\begin{array}{ccc} & & \text{BSO}(k) \\ & \nearrow & \downarrow \\ M & \xrightarrow{\xi} & \text{BSO}(2r+1) \end{array} \quad k = 2s, \quad 2s-1.$$

For (1.1) we have the necessary conditions:

(1.3) (i) mod 2 torsion $P_s(\xi)=0$, $P_{s+1}(\xi)=0$, \dots , $P_r(\xi)=0$;

(ii) the bundle $p^*(\xi)$ over S^n pulls back to a $2s-1$ bundle. (i) follows from the Whitney duality theorem on the integral Pontrjagin classes $P_i(\xi)$. Similarly we have the following necessary conditions for (1.2):

(1.4) (i) mod 2 torsion $P_{s+1}(\xi)=0$, \dots , $P_r(\xi)=0$;

(ii) there exists $u \in H^{2s}(M; \mathbb{Z})$ such that $u^2 = P_s(\xi)$;

(iii) the bundle $p^*(\xi)$ over S^n pulls back to a $2s$ bundle.

(ii) is a necessary condition because in $H^*(\text{BSO}(2s); \mathbb{Z})$ the Pontrjagin class P_s is the square of the Euler class χ_{2s} .

In this paper we make the following assumption:

(1.5) *Hypothesis.* $|\pi|$ = the order of π , is odd.

(1.6) **REMARK.** The case $n=1$ is trivial so we exclude it in all that follows. Also, it is well known that the only group acting freely on an even dimensional sphere is \mathbb{Z}_2 . Therefore $n \geq 3$ is odd and $\pi_1(M) = \pi$. Moreover, according to (2.1), every bundle ξ is a spin bundle (i.e., $w_1(\xi)=0$, $w_2(\xi)=0$). In (1.3) and (1.4) we may drop the condition on two torsions since $H^*(M; \mathbb{Z})$ has only odd torsion.

The following theorems give conditions under which (1.3) and (1.4) are also sufficient.

(1.7) THEOREM. *Let ξ be any $2r+1$ bundle over M . If $q/|\pi|$ is any prime divisor we assume $q > [n/2] - 2s + 2$. Then ξ pulls back to a $2s-1$ bundle if (1.3) is true.*

(1.8) THEOREM. *Again ξ is a $2r+1$ bundle over M . If $q/|\pi|$ is any prime divisor we assume $q > \max([n/2] - s + 1, s + 1)$. Then ξ pulls back to a $2s$ bundle if (1.4) is true.*

(1.9) REMARK. These theorems include the cases where the Pontrjagin classes vanish for dimensional reasons (i.e., $s \geq [n/4]$ in (1.7) and $s > [n/4]$ in (1.8)).

(1.10) REMARK. If one assumes that the action of π on S^n is given by some representation of π by orthogonal matrices then the possible groups π and the possible n have been determined by Vincent (see [11]). In fact, under the restriction that $|\pi|$ is odd, the group π is presented by

$$\sigma^r = 1, \quad \tau^s = 1, \quad \tau\sigma\tau^{-1} = \sigma^t, \quad \text{where } r > 0, |\pi| = rs,$$

$(t-1)s$ is relatively prime to r , and $t^s \equiv 1 \pmod{r}$. Moreover, any such presented group π can act freely and orthogonally on some sphere S^n . The possible values of n are given by easy arithmetic conditions on r, s, t .

In §2 some necessary lemmas are stated, in §3 the proofs are given, and in §4 applications are given. Finally, I would like to thank Professors Spanier and Thomas of Berkeley and the Forschungsinstitut für Mathematik, ETH.

2. Preliminary lemmas. Let C_π denote the Serre class of all finite abelian groups whose orders are relatively prime to $|\pi|$ and let $J \in C_\pi$ be a generic element. We denote by ρ any of the obvious maps $\mathbf{Z} \oplus J \rightarrow \mathbf{Z}$, $\mathbf{Z} \oplus J \rightarrow \mathbf{Z}_{|\pi|}$ = the cyclic group of order $|\pi|$. Then the following is well known:

(2.1) LEMMA. (i) $H^i(M; J) = 0$ for $0 < i < n$ and $|\pi| \cdot H^i(M; \mathbf{Z}) = 0$ for $0 < i < n$, i.e., the order of every element is a divisor of $|\pi|$;

(ii) $\rho_* : H^i(M; \mathbf{Z} \oplus J) \cong H^i(M; \mathbf{Z})$ for $0 < i < n$ and

$\rho_* : H^{2i}(M; \mathbf{Z}) \cong H^{2i}(M; \mathbf{Z}_{|\pi|})$ for $0 < 2i < n$;

(iii) $H^{2i+1}(M; \mathbf{Z}) = 0$ for $0 < 2i+1 < n$;

(iv) $p^* : H^n(M; J) \cong H^n(S^n; J)$ and $p^* : H^n(M; \mathbf{Z} \oplus J) \rightarrow H^n(S^n; \mathbf{Z} \oplus J)$ is a monomorphism.

For the next lemma let $V_{n,n-m}$ be the Stiefel manifold of $n-m$ frames in n -space. Denote by \bar{n} (resp. \bar{m}) the greatest (resp. least) odd integer $\leq n$ (resp. $\geq m$) and denote by S the product $S^{2\bar{m}+1} \times S^{2\bar{m}+5} \times \dots \times S^{2\bar{n}-3}$ where we must include the factor S^{n-1} if n is even and the factor S^m if m is even. If $\bar{n} = \bar{m}$ replace $S^{2\bar{m}+1} \times \dots \times S^{2\bar{n}-3}$ by a point. Putting n_1 equal to the least dimension of all spheres in S we have

(2.2) LEMMA. (i) for all i , $\pi_i(V_{n,n-m})$ and $\pi_i(S)$ are isomorphic mod finite groups.

(ii) $\text{Tor } \pi_i(V_{n,n-m}) \in C_\pi$ if $i < n_1 + 2q - 3$ for all prime divisors $q/|\pi|$ (recall that $|\pi|$ is odd).

Here the E_i , $0 \leq i \leq t$, are the various spaces in the tower and the k_i are the corresponding Postnikov invariants. E and k are the top dimensional space and invariant. Notice that we ignore most of the spaces and invariants in the "actual" Postnikov resolution because the corresponding obstructions are zero as remarked above. This makes no essential difference. Finally the spaces V_i are the fibers of the q_i .

By standard theory we know that ξ lifts up to E_{i+1} in (3.4) if, and only if, there exists a lifting $\eta: M \rightarrow E_i$ so that the obstruction $\eta^*(k_i) \in H^{4s+4i}(M; \pi_{4s+4i-1})$ is zero. By (2.1) $\eta^*(k_i) = 0$ if, and only if, $\eta^* \rho_*(k_i) = 0$. The idea is to relate the obstructions in the groups $H^{4s+4i}(M; \mathbb{Z}_{|\pi|})$ to the Pontrjagin classes $P_{s+i}(\xi)$, $0 \leq i \leq t$, reduced mod $|\pi|$. However, suppose for the moment that ξ has been lifted up to E :

$$\begin{array}{ccccc}
 & & \text{BSO}(2s-1) & & \\
 & & \downarrow & & \\
 & & E & \xrightarrow{k} & K(\pi_{n-1}, n) \\
 & \nearrow \eta & \downarrow & & \\
 S^n & \xrightarrow{p} & M & \xrightarrow{\xi} & \text{BSO}(2r+1)
 \end{array}$$

The last obstruction to lifting ξ is $\eta^*(k) \in H^n(M; \pi_{n-1})$. Now the map $E \rightarrow \text{BSO}(2r+1)$ is a composite of fibrations whose fibers are Eilenberg-MacLane spaces of type $K(G, j)$, $j \leq n-2$. Thus it follows that $\pi_n(E) \cong \pi_n(\text{BSO}(2r+1))$ via the induced map. In particular $\eta \circ p$ is the unique (up to homotopy) lifting of $\xi \cdot p$ up to E . Since $\xi \cdot p$ lifts past E so does $\eta \cdot p$, i.e., $p^* \eta^*(k) = 0$. The homotopy group π_{n-1} is either J or $\mathbb{Z} \oplus J$, where $J \in \mathbb{C}_n$. But in either case (2.1) implies $p^*: H^n(M; \pi_{n-1}) \rightarrow H^n(S^n; \pi_{n-1})$ has zero kernel. Thus $\eta^*(k) = 0$ and ξ lifts up to $\text{BSO}(2s-1)$. In the remainder of the proof we restrict our attention to the k_i .

Since $H^*(\text{BSO}(2r+1); \mathbb{Z}_{|\pi|})$ is a polynomial algebra on the Pontrjagin classes P_1, \dots, P_r we can consider "Pontrjagin classes" $P_{s+i} \in H^{4s+4i}(E_i; \mathbb{Z}_{|\pi|})$ for $s+i \leq r$. Moreover, we would like to "identify" these "Pontrjagin classes" with the invariants $\rho_*(k_i) \in H^{4s+4i}(E_i; \mathbb{Z}_{|\pi|})$ for all $i \leq t$.

Actually, we have

(3.5) LEMMA. $P_{s+i} = 0$ in $H^{4s+4i}(E_{i+1}; \mathbb{Z}_{|\pi|})$ and $= a_i \rho_*(k_i)$ in $H^{4s+4i}(E_i; \mathbb{Z}_{|\pi|})$, where a_i is relatively prime to $|\pi|$.

Proof. By standard Postnikov theory we know that

(3.6) $\pi_j(V_i) = 0$ for $j < 4s+4i-1$ and $f_{i*}: \pi_j(V_i) \cong \pi_j(V_0)$ for $j \geq 4s+4i-1$. By the Serre exact sequence of q_{i+1} it then follows that

$$q_{i+1}^*: H^j(E_{i+1}; \mathbb{Z}_{|\pi|}) \cong H^j(\text{BSO}(2s-1); \mathbb{Z}_{|\pi|})$$

for $j < 4s+4i+3$. In particular $P_{s+i} = 0$ in $H^{4s+4i}(E_{i+1}; \mathbb{Z}_{|\pi|})$ since it is zero in

$H^{4s+4i}(\text{BSO}(2s-1); \mathbb{Z}_{|\pi|})$. Note that this implies P_{s+i} is in the image of the transgression of the fibration q_i and is therefore a multiple of $\rho_*(k_i)$ since

$$H^{4s+4i-1}(V_i; \mathbb{Z}_{|\pi|}) \cong \mathbb{Z}_{|\pi|}$$

is cyclic and $\rho_*(k_i)$ transgresses from the fundamental class of $H^{4s+4i}(E_i; \mathbb{Z}_{|\pi|})$. The lemma for $i=0$ follows directly from a theorem of Borel-Hirzebruch (p. 377 of [1]).

Now consider the commutative diagram of fibrations

$$\begin{array}{ccc} V & \xrightarrow{f} & V' \\ \downarrow & & \downarrow \\ \text{BSO}(2s-1) & \longrightarrow & \text{BSO}(2(s+i)-1) \\ \downarrow & & \downarrow \\ \text{BSO}(2r+1) & \xrightarrow{1} & \text{BSO}(2r+1) \end{array}$$

where $s+i \leq r$, V' is the Stiefel manifold $V_{2r+1, 2r+1-(2(s+i)-1)}$, and all the maps are standard. We use the notation E'_j, k'_j for the right hand fibration in the same way that E_j, k_j is used for the left hand one. According to Borel-Hirzebruch we have $P_{s+i} = \alpha' \rho_*(k'_0)$ in the resolution of the right hand fibration. By naturality of Postnikov resolutions we have the (homotopy) commutative diagram

$$(3.7) \quad \begin{array}{ccccccc} & \vdots & & \vdots & & & \\ & \downarrow & & \downarrow & & & \\ & V_i & \xrightarrow{f \circ f_i} & V' & & & \\ & \downarrow & & \downarrow & & & \\ & \vdots & & \downarrow & & & \\ & V_0 & \xrightarrow{f} & V' & & & \\ & \downarrow & & \downarrow & & & \\ \text{BSO}(2s-1) & \longrightarrow & \text{BSO}(2(s+i)-1) & & & & \\ \downarrow & & \downarrow & & & & \\ q_i & & & & & & \\ E_i & \xrightarrow{g_i} & E'_0 = \text{BSO}(2r+1) & \xrightarrow{\rho_* k'_0} & K(\mathbb{Z}_{|\pi|}, 4s+4i) & & \\ \downarrow & & \downarrow & & & & \\ \text{BSO}(2r+1) & \xrightarrow{1} & \text{BSO}(2r+1) & & & & \end{array}$$

The fiber of $f: V \rightarrow V'$ is the Stiefel manifold $V_{2(s+i)-1, 2t}$ and therefore, by (2.2), we have $f_*: \pi_j(V) \rightarrow \pi_j(V')$ is a C_π isomorphism if $4s+2q-4 > j > 4(s+i)-4$ for all prime divisors q of $|\pi|$. Also from (2.2) we have $\pi_j(V') \in C_\pi$ for $j < 4(s+i)-1$. From (3.6) it now follows that $(f \circ f_i)_*: \pi_j(V_i) \rightarrow \pi_j(V')$ is a C_π isomorphism if $j < 4s+2q-4$ for all prime divisors q of $|\pi|$. From the generalized Whitehead theorem (see (3.3)) and the universal coefficient theorem we get

$$(f \circ f_i)^*: H^j(V'; \mathbb{Z}_{|\pi|}) \rightarrow H^j(V_i; \mathbb{Z}_{|\pi|})$$

is an isomorphism if $j < 4s+2q-5$. The condition $q > [n/2] - 2s + 2$ in (1.7) implies that $4s+4t-1 < 4s+2q-5$ for all such q . Thus

(3.8) $(f \circ f_i)^*: H^{4s+4i-1}(V'; \mathbb{Z}_{|\pi|}) \cong H^{4s+4i-1}(V_i; \mathbb{Z}_{|\pi|})$ for all $i \leq t$. Notice also that both groups are isomorphic to $\mathbb{Z}_{|\pi|}$.

Let τ_i be the transgression in $V_i \rightarrow \text{BSO}(2s-1) \xrightarrow{a_i} E_i$ and let τ' be the transgression in $V' \rightarrow \text{BSO}(2(s+i)-1) \rightarrow E'_0$. There are fundamental classes

$$v_i \in H^{4s+4i-1}(V_i; \pi_{4s+4i-1})$$

and $v' \in H^{4s+4i-1}(V'; \pi_{4s+4i-1}(V'))$. Moreover, the Postnikov invariants

$$k_i \in H^{4s+4i}(E_i; \pi_{4s+4i-1})$$

and $k'_0 \in H^{4s+4i}(E'_0; \pi_{4s+4i-1}(V'))$ are simply $\tau_i(v_i)$ and $\tau'(v')$ resp. By naturality we have $\tau_i(f \circ f_i)^*(\rho_*(v')) = g_i^* \tau'(\rho_*(v')) = g_i^*(\rho_* k'_0)$. Now $P_{s+i} = a' \rho_*(k'_0)$, where a' is relatively prime to $|\pi|$. Thus, in $H^{4s+4i}(E_i; \mathbb{Z}_{|\pi|})$ we have $P_{s+i} = g_i^*(P_{s+i}) = a' g_i^*(\rho_* k'_0) = \tau_i(f \circ f_i)^*(a' \rho_*(v'))$. This together with (3.8) and $\rho_*(k_i) = \tau_i(\rho_*(v_i))$ concludes the proof of (3.5).

Consider now the situation

$$\begin{array}{ccccc} & & E_{i+1} & & \\ & & \downarrow P_{i+1} & & \\ & & E_i & \xrightarrow{k_i} & K(\pi_{4s+4i-1}, 4s+4i) \xrightarrow{\rho} K(\mathbb{Z}_{|\pi|}, 4s+4i) \\ & \nearrow \eta & \downarrow & & \\ M & \xrightarrow{\xi} & \text{BSO}(2r+1) & & \end{array}$$

Suppose $\eta: M \rightarrow E_i$ is some lifting of ξ . The η will lift up to E_{i+1} if, and only if, $\eta^*(\rho_* k_i) = 0$. By (3.5) and (2.1) this happens if, and only if, $\eta^*(P_{s+i}) = 0$. Since $\eta^*(P_{s+i}) = P_{s+i}(\xi)$ and $\rho_*: H^{4s+4i}(M; \mathbb{Z}) \cong H^{4s+4i}(M; \mathbb{Z}_{|\pi|})$ this proves (1.7).

To prove (1.8) consider the lifting problem

$$(3.9) \quad \begin{array}{ccc} V & \longrightarrow & \text{BSO}(2s) \\ & \nearrow \text{dashed} & \downarrow \\ M & \xrightarrow{\xi} & \text{BSO}(2r+1) \end{array}$$

The fiber V is the Stiefel manifold $V_{2r+1, 2r+1-2s}$ and therefore we have

(3.10) (i) $\pi_i(V)$ and $\pi_i(S^{2s} \times S^{4s+3} \times S^{4s+7} \times \dots \times S^{4r-1})$ are isomorphic mod finite groups;

(ii) $\text{Tor } \pi_i(V) \in C_n$ for all $i \leq n-1$.

In the Postnikov resolution of (3.9) we encounter obstructions in the groups $H^{i+1}(M; \pi_i)$, where $\pi_i = \pi_i(V)$, for $i = 2s, 4s-1, 4s+3, 4s+7, \dots, 4s+4t-5, n-1$, where t is chosen to be the largest integer satisfying $4s+4t-5 \leq \min(n-2, 4r-1)$. The hypothesis of (1.8) allows us to conclude, exactly as in the proof of (3.5), that for $i = 4s+3, 4s+7, \dots, 4s+4t-5$, the obstructions to lifting ξ are the Pontrjagin classes $P_{s+1}(\xi), \dots, P_t(\xi)$. However, this is definitely false for $i = 4s-1$. Our Postnikov resolution therefore becomes

$$\begin{array}{ccccccc}
 V & \longrightarrow & \text{BSO}(2s) & & & & \\
 & & \downarrow & & & & \\
 & & E & \xrightarrow{k} & K(\pi_{n-1}, n) & & \\
 (3.11) & & \downarrow & & & & \\
 K(Z, 2s) & \xrightarrow{j} & E_1 & \xrightarrow{k_1} & K(\pi_{4s-1}, 4s) & \xrightarrow{\rho} & K(Z_{|\pi|}, 4s) \\
 & & \downarrow & & & & \\
 M & \xrightarrow[\xi]{} & E_0 = \text{BSO}(2r+1) & \xrightarrow[k_0]{} & K(Z, 2s+1) & &
 \end{array}$$

The first obstruction to lifting ξ is $\xi^*(k_0) \in H^{2s+1}(M; \mathbb{Z})$. By (2.1) this group is zero if $2s+1 \neq n$. If $2s+1 = n$ the obstruction is known to be the Euler class $\chi(\xi)$ (see [9]). But $\chi(\xi) = \delta w_{2s}(\xi)$, where $w_{2s}(\xi)$ is the 2sth Stiefel-Whitney class of ξ and δ is the coboundary for the coefficient sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}_2 \rightarrow 0$. Again (2.1) implies $\xi^*(k_0) = 0$.

The Postnikov invariant k_1 is treated exactly as in the proof of Theorem B in [8]. It follows from [8] that $j^*(\rho_* k_1) = \iota^2$, where $\iota \in H^{2s}(\mathbb{Z}, 2s; \mathbb{Z}_{|\pi|}) \cong \mathbb{Z}_{|\pi|}$ is the fundamental class (here is where we use $q > s+1$ for all prime divisors q of $|\pi|$ for then $H^{4s}(\mathbb{Z}, 2s; \mathbb{Z}_{|\pi|}) \cong \mathbb{Z}_{|\pi|}$ generated by ι^2). As in [8] there is a 1-1 correspondence between liftings $\eta: M \rightarrow E_1$ and cohomology classes $v \in H^{2s}(M; \mathbb{Z})$. Moreover, one has $\eta^*(\rho_* k_1) = v^2 + cP_s(\xi)$ under this correspondence, where c is any integer so that $4c+1 \equiv 0 \pmod{|\pi|}$. Thus ξ lifts past E_1 in (3.11) if, and only if, there exists $v \in H^{2s}(M; \mathbb{Z})$ satisfying $v^2 + cP_s(\xi) = 0$. But this is equivalent to (ii) in (1.4). Finally, the top dimensional obstruction is handled as in the proof of (1.7). This proves (1.8).

4. Applications. Recall that π acts freely on S^n as diffeomorphisms and $M = S^n/\pi$ is the orbit manifold. A theorem of Hirsch (see [2]) states that the manifold M immerses in \mathbb{R}^{n+k} ($k \geq 1$) if, and only if, the stable normal bundle ν pulls back to a k bundle. Then (1.3), (1.4), (1.7) and (1.8) imply

(4.1) THEOREM. (i) Assume $q > [n/2] - 2s + 2$ for all prime divisors q of $|\pi|$. Then M immerses in \mathbb{R}^{n+2s-1} if, and only if, $P_i(\nu) = 0$ for $s \leq i \leq [n/4]$.

(ii) $P_s(\nu) \neq 0$ implies M does not immerse in \mathbb{R}^{n+2s-1} .

(iii) Assume $q > \max([n/2] - s + 1, s + 1)$ for all prime divisors q of $|\pi|$. Then M immerses in \mathbb{R}^{n+2s} if, and only if, $P_i(\nu) = 0$ for $s + 1 \leq i \leq [n/4]$ and there exists $u \in H^{2s}(M; \mathbb{Z})$ such that $u^2 = P_s(\nu)$.

(iv) If there is no $u \in H^{2s}(M; \mathbb{Z})$ such that $u^2 = P_s(\nu)$, then M does not immerse in \mathbb{R}^{n+2s} .

We can also apply our theorems to the tangent bundle τ of M . Any integer $k \geq 1$ can be written uniquely in the form $k = 2^{4a+b}(2c+1)$, where $a, c \geq 0$ and $0 \leq b \leq 3$. Define $\rho(k) = 2^b + 8a - 1$. Also define an r -field to be r vector fields everywhere linearly independent. The following theorem is due to Adams, Hurwitz and Radon:

(4.2) THEOREM. S^{N-1} has a $\rho(N)$ -field but not a $(\rho(N) + 1)$ -field.

After a short calculation one easily proves:

(4.3) LEMMA. (i) for $N \geq 0$ we have $\rho(2N+2) \leq N+4$ with equality only for $N=3$;

(ii) for $N \neq 3$ we have $\rho(2N+2) \leq N+2$ with equality only for $N=1$;

(iii) for $N \neq 0, 1, 3, 7$ we have $\rho(2N+2) \leq N$.

First we consider the special cases. Thus we have

(4.4) THEOREM. (i) all M^3 are parallelizable (indeed, every compact orientable 3-manifold is parallelizable);

(ii) all M^7 admit 4-fields;

(iii) if $3 \nmid |\pi|$, then M^7 has a 5-field if, and only if, there exists $u \in H^2(M; \mathbb{Z})$ satisfying $P_1(M) = u^2$;

(iv) if $3 \nmid |\pi|$, then M^7 is parallelizable if, and only if, $P_1(M) = 0$;

(v) all M^{15} admit 8-fields.

Let ρ equal the maximum number of everywhere linearly independent vector fields on S^n (see (4.2)). Then the general case is

(4.5) THEOREM. (i) $n \neq 3, 7, 15$ implies M^n admits $2[\rho/2]$ -fields (also true for $n=15$ by (v) above). If ρ is even, this is best possible;

(ii) assume $n \neq 3, 7, 15$; ρ is odd; and $q > (n-\rho)/2 + 1$ for all prime divisors q of $|\pi|$. Then M^n admits a ρ -field.

Expressed somewhat differently these theorems say that M^n , for $n \neq 7$, has the same maximal number of linearly independent vector fields as the sphere S^n provided we assume something about $|\pi|$.

(4.6) CONJECTURE. M^n , for $n \neq 7$, has the same number of vector fields as S^n .

Interesting examples of the orbit manifolds M are the lens spaces. Let m be any odd integer and let a_1, \dots, a_{n+1} be integers relatively prime to m . If θ

$= \exp(2\pi(\sqrt{-1})/m)$, then Z_m , the cyclic group presented by: $t, t^m=1$, acts on the sphere S^{2n+1} as follows:

$$t^k(z_1, \dots, z_{n+1}) = (\theta^{ka_1}z_1, \dots, \theta^{ka_{n+1}}z_{n+1}),$$

where (z_1, \dots, z_{n+1}) is a complex $(n+1)$ -tuple representing a point of S^{2n+1} . For the orbit manifold we use the notation $L = L^{2n+1} = L(m; a_1, \dots, a_{n+1})$. The total Pontrjagin class of the tangent bundle τ is given by (see [7]) $P(\tau) = (1 + a_1^2 x^2) \cdots (1 + a_{n+1}^2 x^2)$, where $x \in H^2(L; \mathbb{Z})$ is some class. In particular if all the $a_i = a$ this becomes $P(\tau) = (1 + a^2 x^2)^{n+1}$. Since there is no two torsion in $H^*(L; \mathbb{Z})$ the Whitney duality theorem gives

$$P(\nu) = \sum_{i \geq 0} (-1)^i \binom{n+i}{i} a^{2i} x^{2i},$$

where ν is the stable normal bundle. Thus

$$P_i(\nu) = (-1)^i \binom{n+i}{i} a^{2i} x^{2i}.$$

(4.1) now becomes

(4.7) THEOREM. (i) Assume $q > n - 2s + 2$ for all prime divisors q of m . Then L immerses in \mathbb{R}^{2n+2s} if, and only if,

$$\binom{n+i}{i} \equiv 0 \pmod{m}$$

for $s \leq i \leq [n/2]$.

(ii)

$$\binom{n+s}{s} \not\equiv 0 \pmod{m}$$

implies L does not immerse in \mathbb{R}^{2n+2s} (here $s \leq [n/2]$).

(iii) Assume $q > \max(n-s+1, s+1)$ for all prime divisors q of m . Then L immerses in $\mathbb{R}^{2n+2s+1}$ if, and only if,

$$\binom{n+i}{i} \equiv 0 \pmod{m}$$

for $s+1 \leq i \leq [n/2]$ and

$$(-1)^s \binom{n+s}{s}$$

is a quadratic residue mod m (here $s \leq [n/2]$).

(iv) If

$$(-1)^s \binom{n+s}{s}$$

is not a quadratic residue mod m , then L does not immerse in $\mathbb{R}^{2n+2s+1}$ (here $s \leq [n/2]$).

Using (4.7) together with the following theorem of Olum (see [6]) one can give examples of homotopically equivalent lens spaces which have best possible immersions into different dimensions. Such examples have already been given by Kobayashi.

(4.8) THEOREM. $L(m; a_1, \dots, a_{n+1})$ and $L(m; \lambda_1 a_1, \dots, \lambda_{n+1} a_{n+1})$ are homotopically equivalent if, and only if, there exists an integer k such that $\lambda_1, \dots, \lambda_{n+1} \equiv \pm k^{n+1} \pmod{m}$.

(4.9) EXAMPLE. Take $n=2$ and $m=5$. Then all the lens spaces $L(5; a_1, a_2, a_3)$ are homotopically equivalent. From (4.1) it easily follows that for $(a_1, a_2, a_3) = (1, 1, 1), (1, 4, 1), (1, 1, 4), (1, 4, 4)$ all the corresponding lens spaces do not immerse in \mathbb{R}^7 ; whereas for any other (a_1, a_2, a_3) of the form $(1, a_2, a_3)$ they do.

(4.10) EXAMPLE. Take $n=3$ and $m=7$. Then all lens spaces L^7 are homotopically equivalent. Consider $L=L(7; 2, 1, 1, 1)$ and $L'=L(7; 1, 1, 1, 1)$. Then $P_1(L)=0$ and $P_1(L') \neq 0$. Thus L is parallelizable, where L' is not.

For references on the immersion problem for lens spaces see [3], [4], [5], [8] and [10].

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